The angle property of positive real functions simply derived

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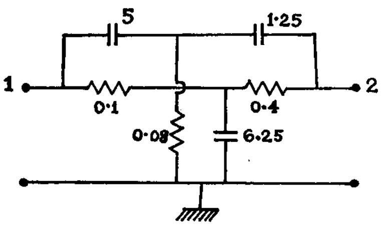

Fig. 2. Realization of the three-terminal RC network for the second-order all-pass transfer function of the example.

The sensitivity of $T(j\omega)$ to circuit elements may be computed as

$$S_A^{T(j\omega)} = 1 \tag{15a}$$

$$S_n^{T(j\omega)} = -\frac{n}{K'}\cdot\frac{D(j\omega)}{D(-j\omega)} = -(n/K')\cos\phi - j(n/K')\sin\phi \tag{15b}$$

and

$$\left|\sum_{i=1}^{m} S_{x_i}^{T(j\omega)}\right| = \left|\sqrt{1+(n/K')^2+2(n/K')\cos\phi}\right|\cdot\left|\sum_{i=1}^{m} S_{x_i}^{T_0(j\omega)}\right| \tag{15c}$$

where x_i is a passive circuit element (a resistor or a capacitor) of the three-terminal RC network, ϕ is the phase function of $T(j\omega)$, and m is the total number of like elements (resistors or capacitors) of the three-terminal network.

The sensitivity of $T(j\omega)$ to the gain of the amplifier has been utilized here in controlling the gain constant of the all-pass function realized. It is seen from (15b) and (15c) that for the network sensitivities to n and the passive elements x_i to be low, the ratio n/K' must be small. For the all-pass realization, the minimum value of the ratio n/K' may be seen to be unity for which

$$\left|\sum_{i=1}^{m} S_{x_i}^{T(j\omega)}\right| = 2\left|\cos\phi/2\right|\cdot\left|\sum_{i=1}^{m} S_{x_i}^{T_0(j\omega)}\right|. \tag{15d}$$

In thin-film networks, the resistances fabricated on the same substrate track closely with temperature, and consequently the variation in the value of n is very small; hence, the gain and phase sensitivities of $T(j\omega)$ to n given in (15b) and having a maximum value of $-n/K'$ each do not pose any problem. It may be noted from (15d) that the sum of the magnitudes of the sensitivities of $T(j\omega)$ with respect to each passive circuit element of the three-terminal network may be minimized by minimizing the sum of the magnitudes of the corresponding sensitivities of $T_0(j\omega)$. As has been indicated in the literature [8], the sum of the sensitivities of the transfer function of a passive three-terminal RC network with respect to each passive element can be made negligible if 1) all the like elements (resistors or capacitors) of the circuit track with temperature, and 2) the fractional variation of a resistor is equal and opposite to that of a capacitor. Thus for low sensitivity figures to passive circuit elements, these conditions must be satisfied, which are not difficult in the thin-film network.

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The Precise Theoretical Limits of Causal Darlington Synthesis

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In this short paper we announce a mathematical theorem concerning Darlington (or Belevitch) synthesis for lumped-distributed circuits. The detailed paper giving proofs will appear in a mathematical journal. A bounded (real) matrix $S(z)$ is said to be (real) losslessly embeddable if there exists a lossless bounded (real) matrix

$$U(z) = \begin{pmatrix} S(z) & A(z) \\ B(z) & C(z) \end{pmatrix}.$$

The theorem we wish to announce is the following.

Theorem: The bounded (real) matrix $S(p)$ is (real) losslessly embeddable if and only if each entry of $S(iw)$ is the boundary value[1] of a meromorphic function[2] defined on the left half-plane.

In particular, $S(p)$ is (real) losslessly embeddable if each entry of $S(p)$ is a rational function, while it is not if one of the entries has an isolated branch point. It is not difficult to see that every lossless bounded matrix $U(p)$ has a meromorphic extension into the left half-plane. The extension can simply be written down; it is $E(p) = [U(-\bar{p})^*]^{-1}$ for each p in the left half-plane. This implies that each entry of $U(p)$ has such a meromorphic extension and, consequently, yields one side of the theorem. The other side of the theorem is much more difficult. A point worth noting is that there are embeddable $S(p)$ [and in fact lossless $S(p)$] which are not actually meromorphic on the entire complex plane. The hypothesis of the theorem does not force $S(p)$ to be meromorphic on the imaginary axis.

Our theorem extends Theorem 4 in Section 6 of DeWilde [1]. He deals with a class of matrix functions which he calls "roomy scattering matrices." Our work shows that these are precisely the matrices, each entry of which is a function analytic in the right half-plane and possessing a meromorphic extension to the left half-plane. We conclude by mentioning that a rather thorough study of functions of this type can be found in [3]. (Although the results there are stated for the unit disk, they hold via conformal mapping for the right half-plane.) One characterization of these functions in terms of rational approximation in [3, theorem 4.1.1.] might be of particular interest.

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[1] $\sigma(iw)$ is the boundary value of the meromorphic function $\nu(z)$ in the sense that $\lim_{x\to 0}\sigma(x+iw) = \nu(iw)$ for almost all w.

[2] We allow only meromorphic functions that can be written as the quotient of bounded analytic functions. For more detail on these definitions, see [4].

The Angle Property of Positive Real Functions Simply Derived

HELGE JÖRSBOE

***Abstract*—The angle property of positive real (rational) functions $Z(s)$, namely, that $|\arg s| \geqq |\arg Z(s)|$ in the right half of the s-plane, can be demonstrated very simply by an examination of the imaginary parts of the functions ln $(s/Z(s))$ and ln $(sZ(s))$, i.e., arg $s \mp$ arg $Z(s)$. In particular, on a contour enclosing the entire first**

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quadrant, arg $s \mp$ arg $Z(s)$ can rather easily be shown to be nonnegative. The extremum theorem of analytic functions then assures that arg $s \mp$ arg $Z(s)$ cannot be negative inside the first quadrant; thus the angle property is demonstrated in the first quadrant. The same result is obtained immediately in the fourth quadrant.

It is an obvious fact that if a function $Z(s)$ is analytic in the right half of the s-plane and if, furthermore,

$$|\arg s| \geqq |\arg Z(s)|$$

in this region, then $Z(s)$ is a positive real function. On the other hand, if a function $Z(s)$ is known to be positive real, then the angle property just stated always holds; this fact, though well known, is far from obvious. In the standard literature two proofs are cited [1], [2], both of which, however, are complex and long winded to the extent that even modern and thorough texts often refrain from presenting a proof; see, e.g., [3], [4]. It is the purpose of this short paper to offer a simple and perspicuous proof of this angle property of positive real functions.

Consider then a positive real (rational) function $Z(s)$, temporarily only in the first quadrant of the s-plane, with a view to proving that in this quadrant $|\arg s| = \arg s \geqq |\arg Z(s)|$ or, equivalently, that both

$$\arg s - \arg Z(s)$$

and

$$\arg s + \arg Z(s)$$

are nonnegative anywhere in this quadrant. The proof will consist of an application of the well-known theorem that, over a region, either component of an analytic function will assume its extremum values on the boundary of that region. The analytic function in question will be $u+jv$, where

$$u + jv = \ln \frac{s}{Z(s)} = \ln \left| \frac{s}{Z(s)} \right| + j(\arg s - \arg Z(s))$$

or

$$u + jv = \ln (sZ(s)) = \ln |sZ(s)| + j(\arg s + \arg Z(s))$$

and it is the imaginary part v of each of these functions that will be examined for its minimum.

In essence, the proof now proceeds as follows. First, an investigation of v is made along a contour enclosing the entire first quadrant. It turns out that v will be nonnegative anywhere on this contour. Next, the theorem just mentioned is applied to the first quadrant including the contour enclosing it. It then becomes evident that

$$v = \arg s \mp \arg Z(s)$$

cannot be negative at any point in this region. So, the angle property has been demonstrated in the first quadrant. Finally, in the fourth quadrant the same result is easily obtained.

Readers interested in an account of the—rather simple—details of the proof are referred to the Appendix.

Appendix

To carry out the proof in precise details, one considers Fig. 1, which shows the first quadrant of the $s = \sigma + j\omega$-plane. Along the contour indicated $v = \arg s \mp \arg Z(s)$ will now be studied. Along the σ-axis, v is evidently zero. At infinity $Z(s)$ must possess either a simple pole with a positive "residue" or a simple zero with a positive "derivative," or it must be equal to a positive constant; to put this more plainly, near infinity $Z(s)$ must behave asymptotically like ρ, Ls, or $1/Cs$, where ρ, L, and C are positive constants. Along the large quarter circle with radius R, then, arg $Z(s)$ will be nearly zero when $Z(s) \simeq \rho$, nearly arg s when $Z(s) \simeq Ls$, and nearly $-$arg s when $Z(s) \simeq 1/Cs$. This means that $v = \arg s \mp \arg Z(s)$ must be nonnegative along the quarter circle with infinite radius $R = \infty$ or, more precisely, that, for any $\epsilon > 0$, however small,

$$v = \arg s \mp \arg Z(s) > -\epsilon$$

will hold along a quarter circle with radius R sufficiently large. To be sure, this estimate is a conservative one. By taking some pains, one could narrow it down to $v \geqq 0$, both here and in the similar cases in the following. However, since this would not facilitate the proof, it is easier to keep the more conservative estimate.

At points on the $j\omega$-axis, $Z(s)$ may possess poles (simple, with positive residues) and zeros (simple, with positive derivatives); in addition, the function s has a zero at the origin. Therefore, either or both

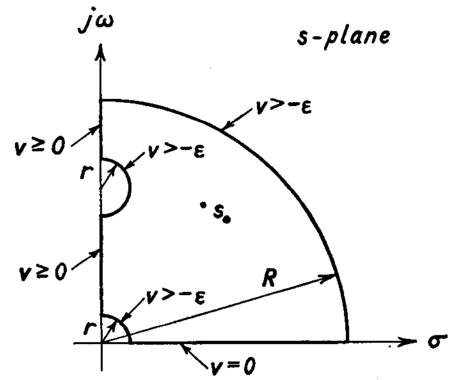

Fig. 1.

of the functions considered here, $\ln (s/Z(s))$ and $\ln (sZ(s))$, will have a singularity at the origin, and at certain other points on the $j\omega$-axis they may both have singularities. In order to secure a closed region of analyticity, the standard artifice of semicircular indentations is applied; at the origin, the indentation will be quarter circular. Now one notices that along the semicircles arg s may get as close to 90° as one desires if only their radii r are made sufficiently small. At the same time, along the semicircles $|\arg Z(s)| \leqq 90°$ since $Z(s)$ is a positive real function. So on infinitely small semicircles $v = \arg s \mp \arg Z(s)$ will be nonnegative.

Along the quarter circle at the origin, a discussion, in fact highly similar to the one given for the large quarter circle, will show that v will be nonnegative on the quarter circle with infinitely small radius $r = 0$.

Along the parts of the contour that are shared with the $j\omega$-axis, v clearly is nonnegative since $\arg s = \arg j\omega = 90°$ and since $|\arg Z(s)| = |\arg Z(j\omega)| \leqq 90°$.

In summary then, along the entire contour one can make $v > -\epsilon$, where $\epsilon > 0$ is arbitrarily small, by sufficiently increasing R and sufficiently decreasing r.

Now suppose that at a finite point s_0 in the first quadrant, either $v = \arg s - \arg Z(s)$ or $v = \arg s + \arg Z(s)$ assumes a negative value. Then, by increasing R and decreasing r sufficiently, one can always make v less negative than this value at all points of the contour. However, according to the theorem mentioned, the lowest value of v must be assumed on the contour. Therefore, the assumption of a negative value of v at some such point s_0 is not possible. So in the first quadrant one has that $v \geqq 0$ or that $|\arg s| \geqq |\arg Z(s)|$.

In the fourth quadrant the same result can be obtained from entirely analogous reasoning or, even simpler, immediately from the identities $|\arg s^*| = |\arg s|$ and $|\arg Z(s^*)| = |\arg Z(s)|$, where $s^* = \sigma - j\omega$ when $s = \sigma + j\omega$. Thus it has been proved that the angle property is a characteristic of all positive real (rational) functions.

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The Maximum Power Transfer Theorem for n-Ports

C. A. DESOER

***Abstract*—The maximum power transfer theorem is proved by an elementary and simple method. The class of all impedance matrices which achieve maximum power transfer is completely described. Cases where $Z_0 + Z_0^*$ is not positive definite are completely discussed.**

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